ELECTROMODELING OF SOLUTIONS OF FOURIER DIFFERENTIAL EQUATIONS

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An examination is made of some aspects of the modeling of heat conduction or diffusion process described by Fourier equations on analog computers. A scheme is given for reproducing a sum of exponential functions with the aid of a single amplifier and a group of switching circuits.

A mathematical description and analysis of a number of heat conduction and diffusion phenomena may be obtained by solving the Fourier differential equation

$$\frac{\partial\Theta}{\partial\tau}=a\nabla^2\Theta \tag{1}$$

with appropriate boundary conditions.

According to the superposition principle, the general solution of (1) may be represented by a sum of particular solutions of the type

$$\Theta(\tau,\ x,\ y,\ z)=\mathrm{A}\varphi(x,\ y,\ z)\exp(-ak^2\tau).$$

In solving applied problems the investigator is especially interested in the variation of a state parameter (temperature, concentration, etc.) with time certain characteristic or investigated points of the system

$$\Theta(\tau)=\sum_{n=0}^{\infty}\mathrm{A}_n\exp(-\alpha_n\tau). \tag{2}$$

When (2) is expressed by a slowly convergent series, it is expedient to use analog computers for analysis of the process. On universal analog computers, the representation of series (2) is accomplished with the help of dc amplifiers operating in the aperiodic link regime (Fig. 1a):

$$U_{\mathrm{ex}}(p)=$$

$$-E\frac{R_0}{R_a}\pm\sum_{i=1}^{n}E\left(\frac{R'_{\mathrm{in}i}}{R'_{\mathrm{in}i}+R''_{\mathrm{in}i}}\right)\frac{R_{0i}}{R_{\mathrm{ex}i}}\left(\frac{1}{1+R_{0i}C_ip}\right)\frac{R_0}{R_i},$$

where

$$U_{\mathrm{ex}}(p)\sim\Theta(\tau),\ E\frac{R_0}{R_a}\sim\mathrm{A}_0,$$

$$E\left(\frac{R'_{\mathrm{in}i}}{R'_{\mathrm{in}i}+R''_{\mathrm{in}i}}\right)\frac{R_{0i}R_0}{R_{\mathrm{ex}i}R_i}\sim\mathrm{A}_i,\ R_{0i}C_i\sim\alpha_i.$$

The initial conditions A_0 and A_i are set up with the help of potentiometers R_{av}, and the exponents α_i—with R_{0i} and R_{exi}. When a series containing n terms has to be modeled, the scheme must consist of n + 1 amplifiers. The presence of a large number of amplifiers and other elements leads to considerable errors in modeling and large power requirements. For this reason a more economic and relatively accurate circuit is proposed containing only one amplifier independently of the number of terms of the series (Fig. 1b). The transfer function of the circuit is described by a similar expression:

$$U_{\mathrm{ex}}(p)= \tag{3}$$

$$-E\frac{R_0}{R_a}\pm\sum_{i=1}^{n}E\left(\frac{R'_{\mathrm{in}i}}{R'_{\mathrm{in}i}+R''_{\mathrm{in}i}}\right)\left[1+\frac{R'_{\mathrm{ex}i}R''_{\mathrm{ex}i}}{R'_{\mathrm{ex}i}+R''_{\mathrm{ex}i}}C_ip\right]^{-1}\frac{R_0}{R_{\mathrm{ex}_i}}.$$

Coefficients A_i and α_i may be varied with the help of potentiometers R_{ini} and R_{exi}. When using standard dc amplifiers whose output voltage drift does not exceed 20 mV and whose input-stage grid current is not more than 10^{-8} A, and with ten terms of the series, the error in reproducing function (2) does not exceed 0.2%.

Resistances R_0 and R_{exi} must not exceed 5 megaohms. The time constant $(R'_{exi}, R''_{exi}/R'_{exi} + R''_{exi})\, C_i$ is determined from the amplitude-frequency characteristics of the dc amplifier in the open-circuit state. Thus, to attain the necessary accuracy of representation (e.g., δ%), it is sufficient that

$$\alpha_i\sim\frac{R'_{\mathrm{ex}i}R''_{\mathrm{ex}i}}{R'_{\mathrm{ex}i}+R''_{\mathrm{ex}i}}C_i\gg\frac{1}{f_{k>2\cdot10^2/\delta}},$$

where f is the frequency at which the gain of the amplifier is equal to $2\cdot10^2/\delta$.

The use of analog computers is of great practical interest in solving inverse problems—the evaluation of thermophysical or diffusion constants from the measured variation of temperature or concentration with time. A technically simpler and more convenient solution of an inverse problem may be accomplished using an automatic scan system based on the method of minimization [1]. The essence of the method is the progressive variation of the variable coefficients and initial conditions of (3) in a direction which reduces the quantity

$$\mu=\sum_{j=1}^{m}|\Theta_j(\tau_j)-\Theta(\tau)|,$$

where Θ_j is the experimental value of the state parameter at discrete times τ_j; $\Theta(\tau)$ is the corresponding

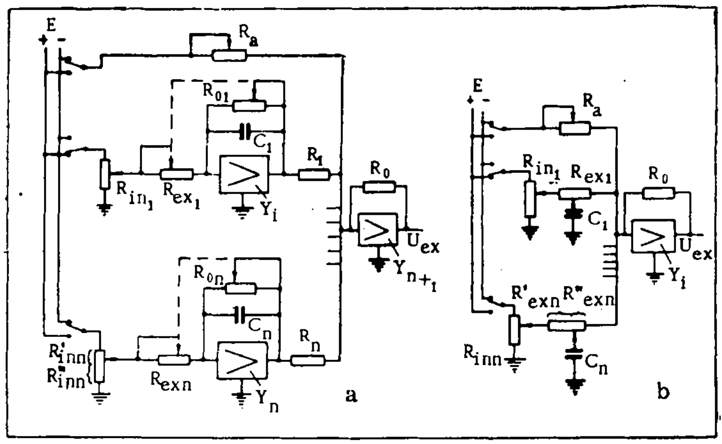

Fig. 1. Functional circuits reproducing series (2): a) circuit using n amplifiers; b) circuit for one amplifier.

value of the simulated state parameter; m is the number of experimental points.

Before the start of the scan, let the variable resistors of the circuit be in any position, corresponding to definite values of coefficients α_{1_0}, α_{2_0}, ..., α_{i_0} and A_{1_0}, A_{2_0}, ..., A_{i_0}. In the search for a solution, one of the coefficients, e.g., α_1, is first varied until the quantity μ attains a minimum value at constant values of the other coefficients. Then the variation of α_1 is stopped, and variation of the next coefficients α_2 is begun, and so on. The process continues until μ is less than some previously assigned value ε, for which the simulated curve of the process coincides to the requisite degree of accuracy with the experimental points. This principle has been used, in particular, to build an automatic scanning system in an analog computer intended for the analysis of kinetic processes [1]. In this machine discrete changes of the coefficients are accomplished by means of voltage dividers and electromechanical step selectors. When the number of variables and the extent of the problems are large, it is expedient to replace the electromechanical switches by electronic ones.

A possible variant of the aperiodic link with electronic switching is shown in Fig. 2. Allowing for the finite values of the reverse and forward resistances R_{rk} and R_{fk} of the switch (Fig. 3a and b), the transfer function of the equipment designed to reproduce (2) will have the form

$$U_{ex}(p) = U_0 \pm \sum_{i=1}^{n} U_{A_i} \frac{R_0}{R_{exi}} \times \left[1 + \frac{R'_{exi}R''_{exi}}{R'_{exi} + R''_{exi}} \left(\frac{R_{rk}C_i p + 1}{R_{rk} + R_{fk} + R_{fk}R_{rk}C_i p}\right)\right]^{-1}.$$

The exact value of the capacitance of the aperiodic link capacitor (Fig. 3b) is determined from the expression

$$C_i = \frac{R_{rk}C_{i\,calc} + R_{fk}C_{i\,calc} - 1}{R_{rk} - R_{rk}R_{fk}C_{i\,calc}}.$$

The dynamic errors of the system when switched to the representation regime (Fig. 3c), due to the different values of the leading edges ($\tau_{\phi 1}$ and $\tau_{\phi 2}$) of

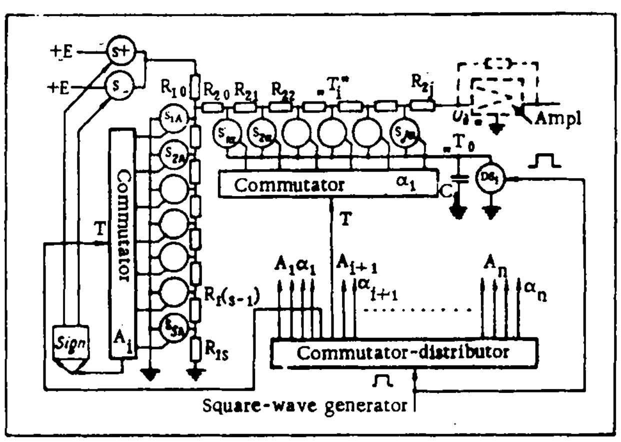

Fig. 2. Aperiodic link of one term of the series with electronic switching.

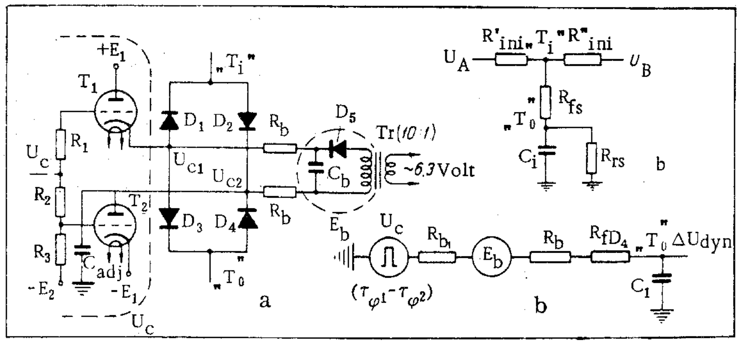

Fig. 3. Diode-triode switching circuit: a) main circuit; b) equivalent circuit of elementary chain and switching elements in the representation regime; c) equivalent circuit of chain for determining dynamic error.

the control voltages U_{c1} and U_{c2}, may be determined from the equation

$$\Delta U_{\text{dyn}} = (E_b + U_{c1})\left[1 - \exp\frac{\tau_{\phi 1} - \tau_{\phi 2}}{2R_b C_i}\right].$$

Thus, with $E_b = 100$ V, $R_b = 50$ kilohms, $U_{c1} = 60$ V, $\tau_{\phi 1} - \tau_{\phi 2} = 2$ millisec, $C_i = 0.02$ μF, and $R_b \gg R_{fD4}$, the error does not exceed 0.1%. Some decrease of this dynamic error may be achieved by adjustment of the leading edges of the control voltages by means of capacitor C_{adj}.

The remaining circuits comprising the automatic solution search system of the analog computer for analysis of kinetic processes (the system for evaluating how close the solution is to the assigned value, the system for choice of direction of coefficient variation, etc.) may be used without any change for solving the problems examined.

To solve inverse problems on analog computers, it is necessary to take into account the possibility of the solution being multivalued, due both to the system assumed for evaluating the correspondence of the simulated curve to the given experimental points, as well as to lack of strict agreement between the laws of the process actually occurring and its mathematical description derived from the idealized boundary problem. Since the order of magnitude of the parameters sought is usually known, it is expedient, in automatic solution search, to limit the range of variation of the parameters to the region of real values for a given concrete problem, using very simple limiting circuits. From the mathematical description of the process, obtained in general form by solving the boundary problem, the ratio between parameters of the process may be established, thus making it possible to choose from a number of sets of coefficients only those which best satisfy the problem examined. In all cases, it is expedient, in the automatic solution search, to limit as much as possible the number of independent coefficients and their range of variation, by introducing into the auto-search process supplementary relationships and limits deriving from the boundary problem solution.

As an example we shall examine the heating of an infinite plate according to the law of convection [2]

$$\frac{\partial \Theta(x, \tau)}{\partial \tau} = a\,\frac{\partial^2 \Theta(x, \tau)}{\partial x^2};$$

$$\tau > 0;\ -r \leqslant x \leqslant +r;\ \Theta(x, 0) = \Theta_0;$$

$$-\lambda\,\frac{\partial \Theta(r, \tau)}{\partial x} + \alpha[\Theta_c - \Theta(r, \tau)] = 0;$$

$$+\lambda\,\frac{\partial \Theta(-r, \tau)}{\partial x} + \alpha[\Theta_c - \Theta(-r, \tau)] = 0.$$

The temperature on the plate surface is determined by the expression

$$\frac{\Theta(r, \tau) - \Theta_0}{\Theta_c - \Theta_0} = 1 - \sum_{n=1}^{\infty} A_n \exp(-\alpha_n \tau), \tag{4}$$

where

$$A_n = \frac{2\sin 2\eta_n}{2\eta_n + \sin 2\eta_n},\quad \alpha_n = \frac{a}{r^2}\,\eta_n^2,$$

and η_n are roots of the equation

$$\eta_n = rh/\mathrm{tg}\,\eta_n.$$

Let it be required to estimate the value of the reduced heat transfer coefficient h according to the experimental curve of surface temperature variation. It follows from the reduced relations that the series (4) is alternating, and, moreover,

$$|A_i| > |A_{i+1}|;\ \alpha_i < \alpha_{i+1}. \tag{5}$$

If several solutions were obtained after this was inserted in the auto-search system, the required solution could be chosen from the additional condition

$$r\sqrt{\alpha_n/a}/\mathrm{tg}\,r\sqrt{\alpha_n/a} = \text{const}. \tag{6}$$

NOTATION

$\Theta(\tau)$—time-dependent state parameter (temperature, concentration, etc.); x, y, z—coordinate variables; λ—thermal conductivity; $Ampl_i$—dc amplifier; S_i—switching circuit; DS_i—discharge switch; R—resistance; C—capacitor; D—semi-conductor diode type D220A; T—triode tube type 6N1P; U_{ex}—output (amplifier) voltage, proportional to the quantity Θ; E—supply voltage; U_c—control voltage; E_b—diode characteristic operating point bias voltage, shaped on capacitor C_b; R_{fD_4}—forward diode resistance.

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